the entrainment into thermals occurs primarily near the rear stagnation point. The path from the rim to the rear stagnation point of a multiburst thermal is long, and it seems likely that the entrainment would be less efficient, i.e., that α would be smaller.

It is hoped that the entrainment constant can be determined experimentally. It should not be difficult to do experiments similar to those that have been done before, but with many small sources rather than one source.

The consequences of the change in cloud geometry have been worked out using the closed-form solution of Ref. 1; but the same change in geometry could be made in a more general analysis and the same general conclusions would be expected.

Acknowlegment

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Similar Solutions for Unsteady Transonic Flow

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Introduction

THIS Note discusses the boundary-value problem for a class of similar solutions in unsteady transonic flow. The formulation introduced here furnishes test cases against which approximate time integration schemes may be checked. Extensions are also outlined for slender body flows.

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Analysis

Exact solutions in transonic flow are rare, and this is particularly true in three-dimensional unsteady problems. The usual idea in fluid mechanics is to reduce the number of dimensions by exploiting various physical symmetries. We shall consider slow temporal accelerations governed by

$$\frac{\partial}{\partial t}(-\Phi_x)+\frac{\partial}{\partial x}\left(K\Phi_x-\frac{1}{2}\Phi_x^2-\Phi_t\right)+\frac{\partial}{\partial y}(\Phi_y)+\frac{\partial}{\partial z}(\Phi_z)=0 \tag{1}$$

where K is the inviscid similarity parameter, Φ is the perturbation potential, and x, y, z and t are the usual independent variables. Separable solutions cannot be obtained in the classical sense because of the transonic nonlinearity. However, the Ansatz $\Phi(x,y,z,t)=A\varphi\ (\eta,\zeta,\xi)$ leads to simplifications if $\eta=x/A$, $\zeta=y/A$, $\xi=z/A$, and A is a function of time. This results in

$$\frac{(K-\varphi_\eta)\varphi_{\eta\eta}+\varphi_{\zeta\zeta}+\varphi_{\xi\xi}}{-2(\eta\varphi_{\eta\eta}+\zeta\varphi_{\eta\zeta}+\xi\varphi_{\eta\xi})}=\frac{\mathrm{d}A(t)}{\mathrm{d}t} \tag{2}$$

Now, in general, the left side is independent of time and the right side is independent of space. It follows that each must be equal to the same constant λ and, further, that $A=A_0+\lambda t$. Equation (2) also defines the boundary-value problem for $\varphi(\eta,\zeta,\xi)$. For subsonic freestreams all disturbances decay far away. Since this must be true for every instant of time, regularity conditions must apply to φ. Neumann boundary conditions on the plane $z=0$ take the form $\Phi_z(x,y,0)=\varphi_\xi(\eta,\zeta,0)$. Thus the normal velocity is prescribed in terms of x/A and y/A.

One final question is the form of the required jump conditions. This is important in shock-fitting analyses. It is answered by recognizing Eq. (1) as the correct small-disturbance expression of mass continuity in proper conservation form.[1] To this we add $[\Phi]=0$, as well as those jump conditions corresponding to $u_y-v_x=0$, $u_z-w_x=0$, and $w_y-v_z=0$ (where $u,v,w=\Phi_{x,y,z}$), all rewritten,[2] of course, in terms of φ. For example, consider two space dimensions x and z. Let $x=f(z,t)$ describe the shock displacement. Then the shock slope (holding time fixed) is

$$\frac{\partial f}{\partial z}=-\frac{[\Phi_z]}{[\Phi_x]}=-\frac{[\varphi_\xi]}{[\varphi_\eta]}$$

while the shock speed (holding z fixed) is

$$\frac{\partial f}{\partial t}=-\frac{[K\Phi_x-\frac{1}{2}\Phi_x^2-\Phi_t]}{[\Phi_x]}-\frac{[\Phi_z]^2}{[\Phi_x]^2}=$$

$$-\frac{[K\varphi_\eta-\frac{1}{2}\varphi_\eta^2+\lambda(\eta\varphi_\eta+\xi\varphi_\xi-\varphi)]}{[\varphi_\eta]}-\frac{[\varphi_\xi]^2}{[\varphi_\eta]^2}$$

Formulas for the three-dimensional case can be written similarly without difficulty. The similarity analysis is easily generalized for slender body flows. The Ansatz used is $\Phi=A\varphi(\eta,\rho,\theta)$, where $\eta=x/A$, $\rho=r/A$ is the radial coordinate, and θ is the azimuthal variable. Putting $\Phi_{rr}+r^{-1}\Phi_r+r^{-2}\Phi_{\theta\theta}$ in Eq. (1) again leads to a separable system, the corresponding boundary conditions being similarly separable.

Discussion

The solutions just introduced describe a restricted class of unsteady flows but they are useful in some limited aerodynamic applications. Because time is eliminated explicitly in the boundary-value formulation, φ can be solved by a modified Murman-Cole algorithm. The results provide a useful source of test cases against which approximate time integration schemes may be checked. It is important that the solution for φ depends on one (prescribed) parameter λ only. Once solved, an infinite number of solutions can be generated directly by adjusting A_0. In this sense, Φ is defined by a two-parameter family of solutions. Because $\mathrm{d}A/\mathrm{d}t=\lambda\neq 0$, the solution would correspond to an initially "steady" solution with an "impulse" at $t=0+$. For $\lambda=0$, of course, our formulation is just the Murman-Cole problem.

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Nonlinear Vibration of Beams with Variable Axial Restraint

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Nomenclature

$A(\xi),A_0$	= area at any station, reference station
$a(\xi)$	= nondimensional area variation
$I(\xi),I_0$	= moment of inertia at any station, reference station
$i(\xi)$	= nondimensional inertia variation
E	= modulus of elasticity
$F(t)$	= time function
$F_p(t),F_q(t)$	= summed up inertial forces at a station
H_B	= spring force
K	= nondimensional spring constant $(=kL/EA_0)$
k	= spring constant
L	= length of beam
M	= bending moment at any station
m_0	= mass per unit length of beam at reference station
N	= axial force at any station
$p(\xi)$	= distributed inertial force
Q	= shear force at any station
$q(\xi)$	= distributed inertial force
V_B	= vertical force at movable hinge
U,V	= displacements of a point on neutral axis in x,y directions
u,v	$=U/L,\ V/L$
x,y	= coordinate system
α	$=V_BL^2/EI_0$
γ	$=H_BL^2/EI_0$
ξ	$=x/L$; nondimensional coordinate
θ,θ_0	= slope at any station, at reference station
λ	= nondimensional quantity $(=m_0\omega^2L^4/EI_0)$
ρ	= radius gyration at reference station
ω	= quantity characterizing vibration

I. Introduction

THE large-amplitude free flexural vibration of beams whose ends are restrained from axial displacement has received considerable attention.[1-5] The geometric nonlinearity is due to the axial force generated by stretching because of the immovability of the supports. The theory in Refs. 1-5 uses linearized curvature expressions and neglects the effect of longitudinal inertia and large deformation. The nonlinear behavior that results always is found to be of a hardening type; i.e., frequency increases with amplitude. A hinged beam, one end of which is immovable and the other movable, i.e., free of axial restraint, also was studied,[6-8] incorporating nonlinearities arising from longitudinal inertia, use of exact curvature expressions, and exact equilibrium equations (i.e., specification of loads in terms of the deformed configuration). The behavior in this case is of the softening type. Wrenn and Mayers[9] studied a general case of variable axial restraint, introducing the nonlinearity due to axial force alone. Thus, for the limiting case of infinite axial restraint, their results agree with those of beams on immovable supports,[1-5] whereas, for zero axial restraint, they obtain zero nonlinearity, contrary to the results reported in Refs. 6-8. Clearly, such a formulation must include the effects of longitudinal inertia and large curvatures. This Note therefore considers this general case, using a formulation in two displacement quantities developed in Ref. 10 and reported in Refs. 11 and 12.

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II. Theory

From Fig. 1, the force equilibrium at station ξ is expressed as

$$N+F_p(\xi)\cos\theta+F_q(\xi)\sin\theta+H_B\cos\theta=V_B\sin\theta \tag{1}$$

$$Q+F_q(\xi)\cos\theta-F_p(\xi)\sin\theta-H_B\sin\theta=V_B\cos\theta \tag{2}$$

$$M,x-(1+\bar{\epsilon})Q=0 \tag{3}$$

where $F_p(\xi)$, $F_q(\xi)$ are the inertial forces summed up as

$$F_p(\xi)=L\int_\xi^1 p(\xi)\,\mathrm{d}\xi,\quad F_q(\xi)=L\int_\xi^1 q(\xi)\,\mathrm{d}\xi$$

and $\bar{\epsilon}$ is defined below. The differential equation of motion is obtained from Eqs. (2) and (3) as[10]

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[i(\xi)\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right]+(1+\bar{\epsilon})\frac{L^2}{EI_0}\left[F_p(\xi)\sin\theta-F_q(\xi)\cos\theta\right.$$
$$\left.+V_B\cos\theta+H_B\sin\theta\right]=0 \tag{4}$$

and the strain quantities $\epsilon=N/EA_0$, $\epsilon_0=V_B/EA_0$, and $\bar{\epsilon}=N/EA(\xi)$ are related as

$$\bar{\epsilon}=\frac{\epsilon}{a(\xi)}=\frac{\epsilon_0\sin\theta}{a(\xi)}-\frac{F_p(\xi)}{EA(\xi)}\cos\theta-\frac{F_q(\xi)}{EA(\xi)}\sin\theta$$
$$-\frac{H_B}{EA(\xi)}\cos\theta \tag{5}$$

The analysis is simplified by assuming that $\bar{\epsilon}\ll 1$ within the elastic limit. Therefore, without much loss of accuracy, we have

$$U(\xi,t)=L\int_0^\xi(\cos\theta-1)\,\mathrm{d}\xi \tag{6}$$

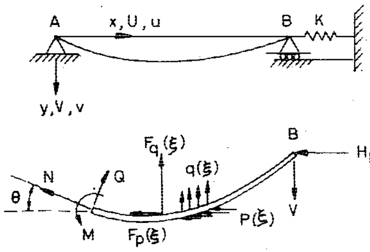

Fig. 1 Initial configuration and free body diagram at position of maximum amplitude.